VISCOUS DISSIPATION DURING HEAT TRANSFER IN A PIPE

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Dissipation of mechanical energy plays an important role in many heat transfer processes in highly viscous materials. A proper understanding of this phenomenon and its consequences for pipe flow might be based on the solution of the heat transfer problem at a simplified assumption of fully developed and temperature-independent velocity profile. We present an exact solution of this problem obtained by the method of separated variables represented by a series of eigenfunctions simultaneoulsy with an asymptotic similarity solution for the region of low values of z where the convergence of the series is slow.

Increase in the enthalpy of liquid during the pipe flow, which is brought about by dissipation, is directly proportional to pressure loss. It depends on the length and diameter of the pipe whether this dissipation-induced increase in temperature will be considerable. A substantial pressure loss has been often observed even in short pipe sections ($z \ll 0{\cdot}1$) which is usual for *e.g.* injector and extruder dies used in polymer processing, inlet systems of combustion chambers for fuel suspensions and several other technological processes working with highly viscous liquids at high pressures. In such cases, the dissipative heat evolved usually in a thin layer at the wall cannot be distributed, which may lead to increase in the temperature of the layer by as much as tens or hundreds of degrees.

Heat transfer processes with significant viscous dissipation are usually modelled by systems of partial differential equations; at simplifying assumptions, several numerical approximations for high Prandtl numbers may be found in the literature[1-4,13]. If we are attempting to find an analytical solution, we must introduce further simplifications. The assumption of a developed temperature profile[5], which simplifies axial coordinate dependences of the velocity and temperature but which simultaneously yields results suitable for long pipes only, may serve as an example of such an approximation applied to dissipative heat transfer. The case which is dealt with in our work is based on the assumption of temperature-independent viscosity, which makes it possible to solve momentum and heat transfer equations independently. A solution of the temperature field for this case has been given by Toor[6]; his method has been extended to a wide variety of boundary conditions by Petukhov[7]. The significance of this problem, which was being overlooked as a rather crude approximation, has been injustly underestimated. Its solution gives exact model laws for the idealized case of a negligible temperature dependence of viscosity, however, we believe that these results may be used as a guide for an extrapolation of separate numerical results obtained from the solution of the complete mathematical model. These laws may also serve as a useful basis for a formulation of semiempi·ical correlations of a larger set of results yielded by a rigorous numerical solution[8]. One of the reasons preventing the analytical solution from becoming widespread in the applications was clumsiness of its results expressed in the form of a series of eigenfunctions and especially the fact that convergence of this series is slow in the region of low values of z, which is important in equipments for polymer processing. Due to this fact we present in our work besides the complete standard analytical solution also a simple asymptotic formula approximating this solution for $z \to 0$ and we will show the limits of its applicability.

Mathematical Model

For a steady-state pipe flow of a non-Newtonian fluid obeying the temperature--independent power-law model

$$\tau = K(-\mathrm{d}v/\mathrm{d}R)^n \tag{1}$$

the solution of the momentum balance is given by the velocity profile, which may be substituted into the equation for heat transfer; this equation assumes then the form of

$$\varrho C_p U \frac{3n+1}{n+1}\left[1 - \left(\frac{R}{R_1}\right)^{1+1/n}\right]\frac{\partial T}{\partial x} = \frac{k}{R}\frac{\partial}{\partial R}\left(R\frac{\partial T}{\partial R}\right) + K\left[\frac{(3n+1)}{n}\frac{U}{R_1}\right]^{1+n}\left(\frac{R}{R_1}\right)^{1+1/n}. \tag{2}$$

Usual boundary conditions of the first kind

$$T = T_0 \quad \text{for} \quad x = 0\,, \tag{3}$$

$$T = T_w \quad \text{for} \quad x \geqq 0 \text{ and } R = R_1\,, \tag{4}$$

$$\partial T/\partial R = 0 \quad \text{for} \quad R = 0\,, \tag{5}$$

allow us to write the complete solution as the sum of two terms

$$T = T_1 + T_{1D}\,, \tag{6}$$

where T_1 is solution T of Eq. (2) with boundary conditions (3)–(5) on the assumption that $K = 0$ (without the dissipation term); T_{1D} is solution T of Eq. (2) at boundary conditions (5) and

$$T = T_0 \quad \text{for} \quad x = 0\,, \tag{7}$$

$$T = T_0 \quad \text{for} \quad R = R_1\,. \tag{8}$$

On introducing dimensionless variables

$$r \equiv R/R_1\,, \quad z \equiv xk/(\varrho c_p U R_1^2) \qquad (9), (10)$$

and quantity T_D with the dimension of temperature:

$$T_D \equiv 2(3 + 1/n)^n \,.\, K U^{1+n} R_1^{1-n}/k\,, \tag{11}$$

Eq. (2) may be rewritten into the form

$$\frac{3n + 1}{n + 1}\left(1 - r^{1+1/n}\right)\frac{\partial T}{\partial z} = \frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{3n + 1}{2n} T_D r^{1+1/n}\,, \tag{12}$$

where quantities with the dimension of temperature are retained because we have at our disposal no quantity with this dimension which would be simultaneously suitable for normalization.

Only separate increments in superposition (6) can be purposefully expressed through dimensionless functions. The first increment as given by the effect of boundary conditions is equal to

$$T_1(r, z) = T_0 + (T_w - T_0)(1 - t_1(r, z))\,, \tag{13}$$

where t_1 is the solution of

$$\frac{3n + 1}{n + 1}\left(1 - r^{1+1/n}\right)\frac{\partial t_1}{\partial z} = \frac{\partial^2 t_1}{\partial r^2} + \frac{1}{r}\frac{\partial t_1}{\partial r}\,, \tag{14}$$

with boundary conditions

$$t_1 = 1 \quad \text{for} \quad z = 0\,, \tag{15}$$

$$t_1 = 0 \quad \text{for} \quad z \geqq 0 \text{ and } r = 1\,, \tag{16}$$

$$\partial t_1/\partial r = 0 \quad \text{for} \quad r = 0. \tag{17}$$

This solution, already known for the dissipation-free heat transfer[9,10], may be expressed through eigenfunctions Y_i and eigenvalues b_i^2 corresponding to the set of equations

$$\frac{d^2 Y_i}{dr^2} + \frac{1}{r}\frac{dY_i}{dr} + b_i^2 \left(\frac{3n+1}{n+1}\right)\left(1 - r^{1+1/n}\right) Y_i = 0 \qquad (18)$$

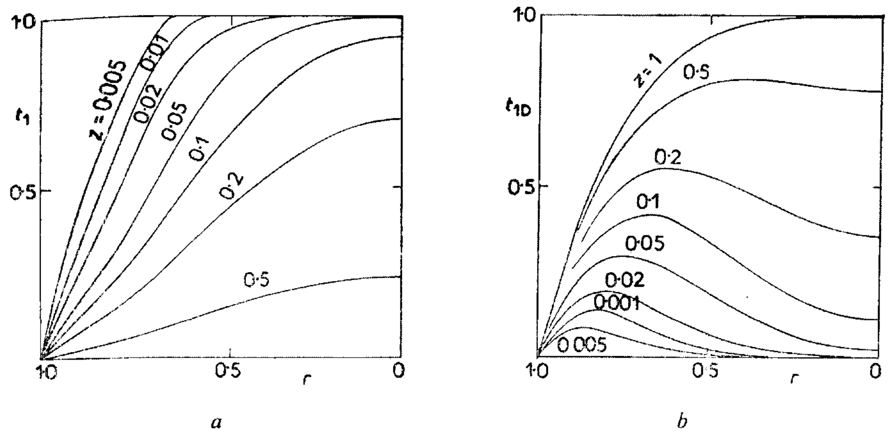

FIG. 1

Dimensionless Temperature Fields t_1 and t_{1D} for a Newtonian Fluid ($n = 1$)

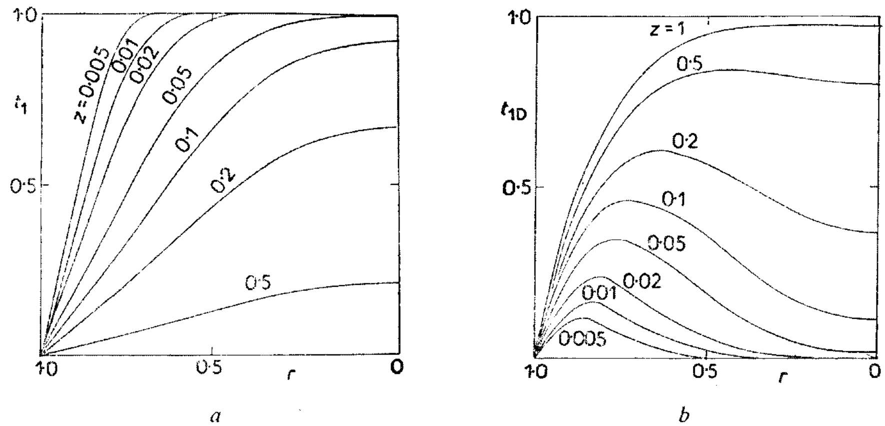

FIG. 2

Dimensionless Temperature Fields t_1 and t_{1D} for a Non-Newtonian Fluid with the Flow Index $n = 0.5$

and boundary conditions

$$Y_i(1) = 0\,, \tag{19}$$

$$\mathrm{d}Y_i/\mathrm{d}r = 0 \quad \text{for} \quad r = 0\,. \tag{20}$$

It holds

$$t_1(r, z) = \sum_{i=1}^{\infty} \frac{\int_0^1 Y_i(r)\,(r - r^{2+1/n})\,\mathrm{d}r}{\int_0^1 Y_i^2(r)\,(r - r^{2+1/n})\,\mathrm{d}r}\, Y_i(r) \exp\left[-b_i^2 z\right]. \tag{21}$$

Convergence of this series at high values of z is good.

We express increment T_{1D} corresponding to the dissipation as

$$T_{1D} = T_D\,.\,(1/2)n/(3n + 1)\,.\,t_{1D} \tag{22}$$

where the dimensionless function $t_{1D}(r, z)$ is the solution of

$$\frac{3n+1}{n+1}\left(1 - r^{1+1/n}\right)\frac{\partial t_{1D}}{\partial z} = \frac{\partial^2 t_{1D}}{\partial r^2} + \frac{1}{r}\frac{\partial t_{1D}}{\partial r} + \left(\frac{3n+1}{n}\right)^2 r^{1+1/n}\,, \tag{23}$$

at the conditions

$$t_{1D} = 0 \quad \text{for} \quad z = 0\,, \tag{24}$$

$$t_{1D} = 0 \quad \text{for} \quad r = 1\,, \tag{25}$$

$$\partial t_{1D}/\partial r = 0 \quad \text{for} \quad r = 0\,. \tag{26}$$

This solution can be expressed through eigenfunctions of system (18)–(20) as

$$t_{1D}(r, z) = 1 - r^{3+1/n} - \sum_{i=1}^{\infty} \frac{\int_0^1 (1 - r^{3+1/n})\, Y_i(r)\,(r - r^{2+1/n})\,\mathrm{d}r}{\int_0^1 Y_i^2(r)\,(r - r^{2+1/n})\,\mathrm{d}r}\, Y_i(r) \exp\left[-b_i^2 z\right]. \tag{27}$$

The eigenvalues and eigenfunctions were calculated by the method described in an earlier work[10]. The quadratures were executed also in formulas (21) and (27). The temperature field during dissipative heat transfer may be therefore constructed by the relation

$$T(r, z) = T_0 + (T_w - T_0)\left[1 - t_1(r, z)\right] + \tfrac{1}{2}n/(3n + 1)\, T_D\,.\, t_{1D}(r, z)\,, \tag{28}$$

and employing the knowledge of the field of functions $t_1(r, z)$ and $t_{1D}(r, z)$ recorded for different values of flow index n on Figs 1–3.

Asymptotic Solution of Functions t_1 and t_{1D} for Low Values of $z \to 0$

Both series (*21*) and (*27*) converge slowly at small z. (For illustration let us note that the series of 10 terms is necessary to achieve a 95% accuracy at $z = 0{\cdot}001$.) As it is obvious from Figs 1–3, the heat transfer at small z proceeds only in the immediate vicinity of the wall, where some simplifying assumptions may be accepted. On introducing the dimensionless distance from the wall

$$y \equiv 1 - r\,, \tag{29}$$

then at $y \to 0$ relation (*14*) reduces to

$$\frac{3n+1}{n}\, y \frac{\partial t_1}{\partial z} = \frac{\partial^2 t_1}{\partial y^2}\,. \tag{30}$$

It is known[11] that both Eq. (*30*) and boundary conditions (*15*)–(*17*) are satisfied by the ordinary differential equation

$$\frac{\partial^2 t_1}{\partial^2 \xi} + \frac{\xi^2}{3} \frac{\partial t_1}{\partial \xi} = 0 \tag{31}$$

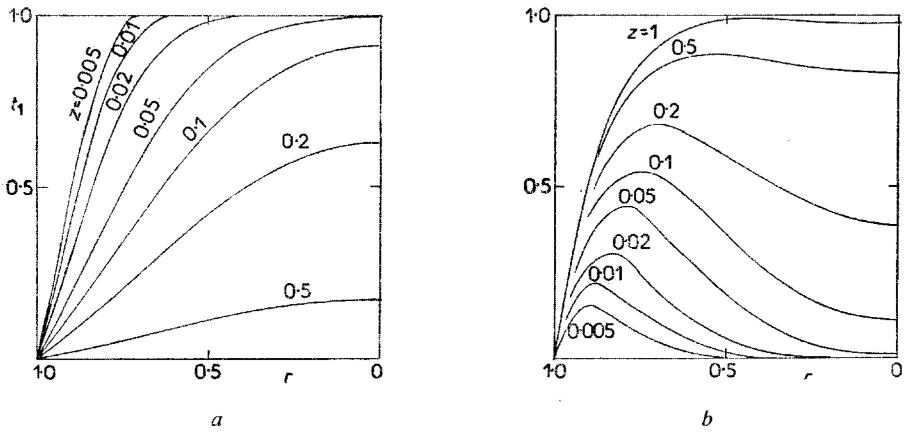

Fig. 3

Dimensionless Temperature Fields t_1 and t_{1D} for a Non-Newtonian Fluid with the Flow Index $n = 0{\cdot}25$

with the boundary conditions

$$t_1 = 0 \quad \text{for} \quad \xi = 0 \tag{32}$$

$$t_1 = 1 \quad \text{for} \quad \xi \to \infty \tag{33}$$

and

$$\xi \equiv y[(3n + 1)/(zn)]^{1/3}\,. \tag{34}$$

Fig. 4 shows that the solution of Eq. (*31*) is a good approximation of the solution of the complete system (*14*)–(*17*) for $z < 0.1$.

Eq. (*23*) at small values $y \to 0$ reduces to

$$\frac{3n + 1}{n}\, y\, \frac{\partial t_{1D}}{\partial z} = \frac{\partial^2 t_{1D}}{\partial r^2} + \left(\frac{3n + 1}{n}\right)^2. \tag{35}$$

It can be shown that the solution of the ordinary differential equation

$$\frac{d^2\psi}{d\xi^2} + \frac{\xi^2}{3}\frac{d\psi}{d\xi} - \tfrac{2}{3}\xi\psi + 1 = 0\,, \tag{36}$$

with the boundary conditions

$$\psi = 0 \quad \text{for} \quad \xi = 0\,, \tag{37}$$

$$\psi = 0 \quad \text{for} \quad \xi \to \infty\,, \tag{38}$$

satisfies after substituting (*34*) and (*39*)

$$t_{1D} = [(3n + 1)/n]^{4/3}\, z^{2/3}\psi(\xi) \tag{39}$$

the differential equation (*35*) and conditions (*24*)–(*26*) for $z \to 0$. It follows from Fig. 5, where this solution is compared with the exact solution of complete Eq. (*23*), that the approximate solution reflects the character of profile $t_{1D}(y)$, indicates the maximal value of t_{1D} at $\xi \approx 1.3$ in accordance with the exact solution, but nevertheless it differs in the absolute values. This may be due to approximating the dissipative term by a value at the wall in passing from Eq. (*23*) to (*35*) and having in mind the solubility of the equation. A certain correction might be achieved if we assume that the dissipation is controlled by the velocity gradient in the region of the maximal temperature, where $y_D \approx 1.3[zn/(3 + 1)]^{1/3}$; then Eq. (*35*) with the dissipation term multiplied by factor $(1 - y_D)^{1+1/n}$ has the solution

$$t_{1D}^* = \left(\frac{3n+1}{n}\right)^{4/3} z^{2/3} \left[1 - 1\cdot3\left(\frac{zn}{3n+1}\right)^{1/3}\right]^{1+1/n} \psi(\xi) \tag{40}$$

which, as it is obvious from Fig. 5, approximates the exact solution better for $z < 0\cdot1$.

For the heat transfer in a pipe with a constant wall temperature T_w and significant dissipation term, the temperature field in the temperature inlet section may be described by relation (*28*) into which asymptotic expressions for t_1 and t_{1D} obtained by solving Eqs (*30*) and (*35*) have been inserted:

$$\lim_{z\to 0} T(\xi, z) = T_0 + (T_w - T_0)\left[1 - t_1(\xi)\right] + (3 + 1/n)^{1/3+n} \,.$$
$$. \, KU^{1+n}R_1^{\,-n}/k \; z^{2/3} \, \psi(\xi) \tag{41}$$

Based on these relations, an approximate formula with correction (*40*) can be derived

$$T(\xi, z) \approx T_0 + (T_w - T_0)\left[1 - t_1(\xi)\right] + (3 + 1/n)^{1/3+n} \,.\, KU^{1+n}R_1^{1-n}/k \,.$$
$$. \, z^{2/3}\{1 - 1\cdot3[zn/(3n+1)]^{1/3}\}^{1+1/n} \,.\, \psi(\xi)\,, \tag{42}$$

which is valid within the accuracy of technical calculations for $z < 0\cdot1$.

Because functions $\psi(\xi)$ and $1 - t_1(\xi)$ are approximately of the same order, the criterion may be introduced

$$\Pi_{D0} \equiv \mathrm{Br}(3 + 1/n)^{1/3+n} \, z^{2/3}\,, \tag{43}$$

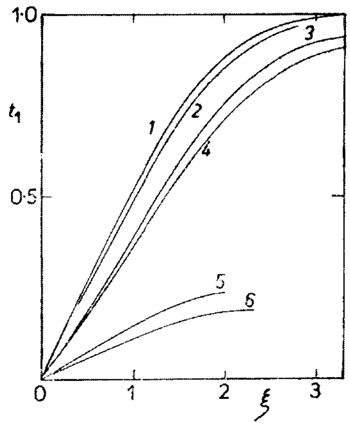

FIG. 4

Comparison of the Asymptotic Solution of Function t_1 for $z \to 0$ (curve *1*) with Actual Courses of the Function for Finite Values of z: *2* for $z = 0\cdot01$ and $n = 1$, *3* for $z = 0\cdot1$ and $n = 1$, *4* for $z = 0\cdot1$ and $n = 0\cdot333$, *5* for $z = 0\cdot5$ and $n = 1$, *6* for $z = 0\cdot5$ and $n = 0\cdot5$

where

$$\mathrm{Br} \equiv KU^{1+n}R_1^{1-n}/(k\,|T_w - T_0|) \tag{44}$$

is the Brinkmann number defined in the usual manner. The significance of the effect of dissipation on the temperature field for $z < 0{\cdot}1$ is then given by criterion Π_{D0}.

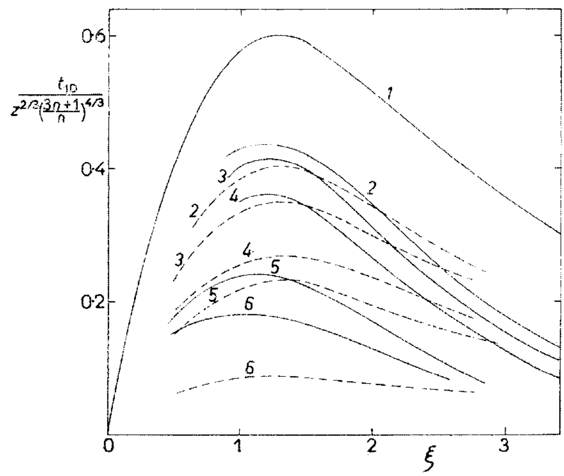

FIG. 5

Comparison of the Asymptotic Solution of Function t_{1D} for $z \to 0$ (Curve *1*) with Actual Courses of the Function at Finite Values of z: Curve *2* for $z = 0{\cdot}01$ and $n = 1$, *3* for $z = 0{\cdot}01$ and $n = 0{\cdot}5$, *4* for $z = 0{\cdot}01$ and $n = 0.25$, *5* for $z = 0{\cdot}1$ and $n = 1$, *6* for $z = 0{\cdot}1$ and $n = 0{\cdot}25$

Dashed are approximations employing the corrected formula (*40*) and corresponding to cases *2*–*6*.

It is seen on Fig. 3 that functions $(1 - t_1)$ and t_{1D} are approximately of the same order for $z > 0{\cdot}1$, so that the criterion of significance of dissipation in this region following from Eq. (*28*) is given by

$$\Pi_{D\infty} = \mathrm{Br}(3 + 1/n)^{n-1}\,, \tag{45}$$

which is a modification of the Brinkman number suitable for non-Newtonian fluids.

CONCLUSIONS

1) A solution of the partial differential equation for heat transfer during the inlet of a power-law fluid with a temperature-independent consistency and temperature T_0 into a pipe with the wall temperature T_w and under the influence of dissipation was formulated in expression (*28*), which separates the effects of boundary conditions and dissipation. Courses of functions $t_1(r, z)$ and $t_{1D}(r, z)$ appearing in this formula are illustrated on Figs 1–3. For low values of $z < 0{\cdot}1$, the corrected asymptotic formula (*42*) with functions $t_1(\xi)$ and $\psi(\xi)$ depicted on Figs 4 and 5 may be recommended.

2) The effect of more complicated boundary conditions imposed *e.g.* by a variable temperature of the entering liquid or by the axially-dependent wall temperature, will be reflected in expression (*28*) only through function $t_1(r, z)$. Starting from the knowledge of the fundamental solution given by expression (*21*), this function can be evaluated by the superposition principle[12,13] without necessarily solving the differential equations once again.

3) The effect of the dissipation on heat transfer was evaluated and it may be stated that this effect is measured at $z < 0{\cdot}1$ by criterion Π_{D0} or by criterion $\Pi_{D\infty}$ at $z > 0{\cdot}1$. Quantitatively it can be stated that the dissipation plays the major role while temperature changes at the wall are of minor importance

$$\text{at} \quad z < 0{\cdot}1 \quad \text{and} \quad z \gg \left(\frac{n}{3n + 1}\right)^{(3n+1)/2} / \mathrm{Br}^{3/2} \tag{46}$$

$$\text{and} \quad \text{at} \quad z > 0{\cdot}1 \quad \text{and} \quad \mathrm{Br}(3 + 1/n)^{n-1} \gg 1\,. \tag{47}$$

The contribution to the temperature field due to the dissipation is unimportant

$$\text{at} \quad z < 0{\cdot}1 \quad \text{and} \quad z \ll \left(\frac{n}{3n + 1}\right)^{(3n+1)/2} / \mathrm{Br}^{3/2} \tag{48}$$

$$\text{and} \quad \text{at} \quad z > 0{\cdot}1 \quad \text{and} \quad \mathrm{Br}(3 + 1/n)^{n-1} \ll 1\,. \tag{49}$$

It may be assumed that at conditions (*48*) or (*49*), findings on the dissipation-free heat transfer can be applied even to systems for which the heat transfer equation deviates from Eq. (*2*) – *e.g.* for other rheological models or for a flow with a temperature-dependent viscosity; it is only the matter of suitable definitions of characteristic values K and n.

LIST OF SYMBOLS

b_i^2 eigenvalues of system (*18*)–(*20*)
c_p specific heat
Br Brinkman number (*44*)
k thermal conductivity
K consistency coefficient (*1*)
n flow index (*1*)

R radial coordinate
R_1 radius of the pipe
r dimensionless radial coordinate (*9*)
T temperature
T_D characteristic dissipation temperature (*11*)
T_1 solution of the heat transfer equation without dissipation
T_{1D} solution of the heat transfer equation without the temperature step on the wall
T_0 temperature of the entering liquid
T_w wall temperature
t_1 dimensionless temperature (*13*)
t_{1D}^* dimensionless temperature (*22*)
t_{1D} approximate solution of function t_{1D} (*40*)
U mean velocity
x axial coordinate
Y_i eigenfunction of system (*18*)–(*20*)
y dimensionless distance from the wall (*29*)
z dimensionless axial coordinate (*10*)
ξ dimensionless variable (*34*)
Π_{D0} criterion of the effect of dissipation at $z \rightarrow 0$ (*43*)
$\Pi_{D\infty}$ criterion of the effect of dissipation at $z \rightarrow \infty$ (*45*)
ϱ density
ψ dimensionless function (*39*)

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